Machine Learning Workshop, 11/29/2016

Exact machine learning topological phases with an artificial neuron network

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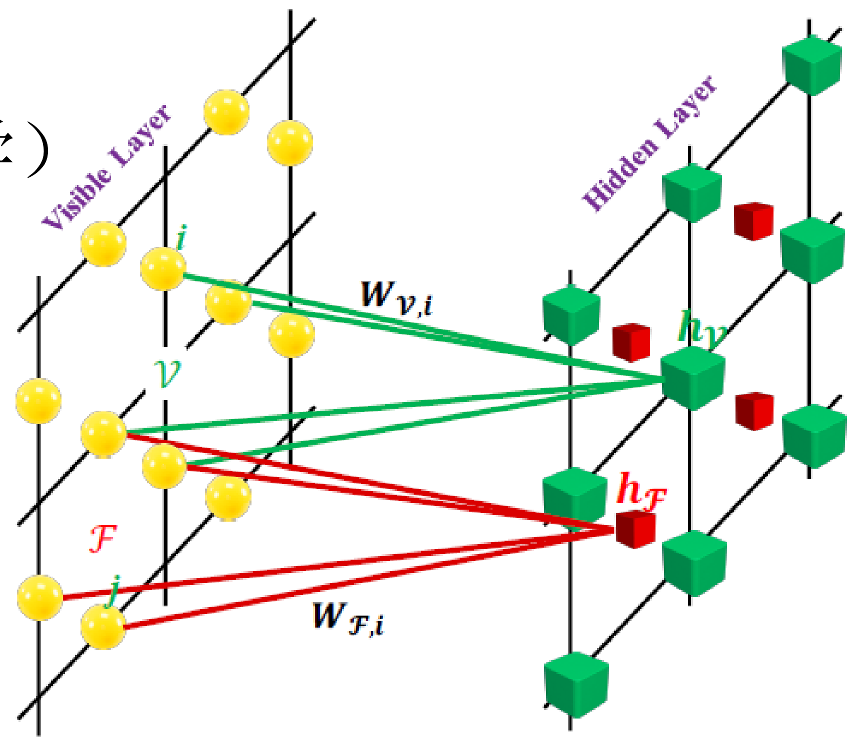
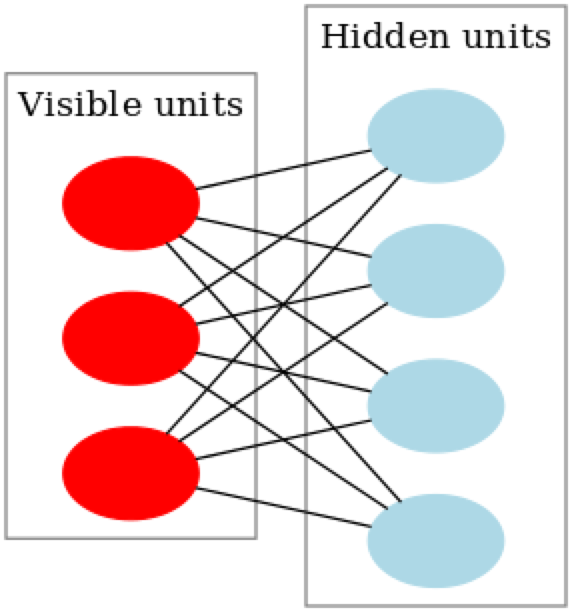
Dong-Ling Deng et al., arXiv (2016)



Outline

- Restricted Boltzmann machine and a quantum neuron network wavefunction
- Exact representation of 1d cluster state with the QNNW
- Exact representation of 2d Toric model with the QNNW

Restricted Boltzmann Machine (RBM)



v_i h_j

$$P(v) \propto \sum_h \exp\left[\sum_i a_i v_i + \sum_j b_j h_j + \sum_{ij} W_{ij} v_i h_j\right]$$

- **no intra-layer coupling**
- **decoupled when h is fixed**
- **easy to sample (no need for Markov-chain sampling, no autocorrelation, …)**

$$P(h) \propto \exp\left[\sum_j b_j h_j\right]$$

$$P(v \mid h) \propto \exp\left[\sum_i a_i v_i + \sum_{ij} W_{ij} v_i h_j\right]$$

Quantum version of RBM

Replacing the probability by a QNN wave function

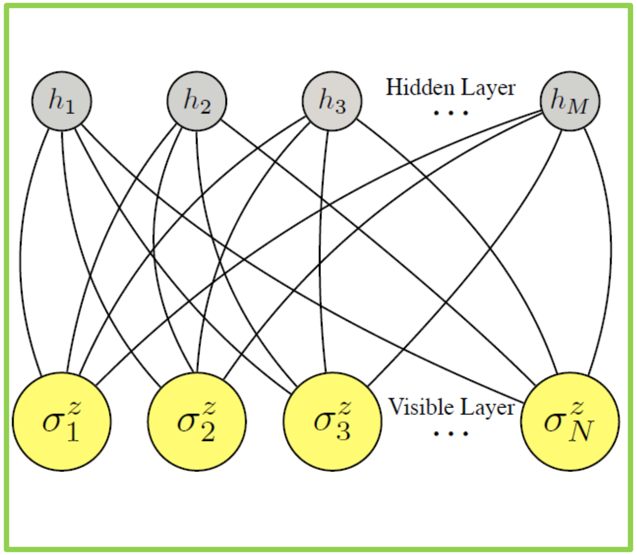
$$\Psi_M(\mathcal{S};\mathcal{W}) = \sum_{\{h_i\}} e^{\sum_j a_j \sigma_j^z + \sum_i b_i h_i + \sum_{ij} W_{ij} h_i \sigma_j^z}$$

Examples for quantum models:

Transverse field Ising: $\mathcal{H}_{\mathrm{TFI}} = -h \sum_i \sigma_i^x - \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$

Anti-Ferro Heisenberg: $\mathcal{H}_{\mathrm{AFH}} = \sum_{\langle i,j \rangle} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z$

G. Carleo and M. Troyer, arXiv: 1606.02318 (2016)



Open Questions for QNN wavefunction

- Entanglement properties
- How to implement non-abelian symmetries (to combine convolutional network)
- Redundancy and optimal representation
- Exact non-trivial examples (analogous to AKLT state for MPS)

1d SPT cluster state

Hamiltonian: $$H_{\text{cluster}} = -\sum_{k=1}^{N} \hat{\sigma}^z_{k-1}\hat{\sigma}^x_k\hat{\sigma}^z_{k+1}$$

$Z_2 \times Z_2$ symmetry: $\hat{\sigma}^{z,y} \rightarrow -\hat{\sigma}^{z,y}$ for all even or all odd sites

Stabilizer ground state: $\hat{\sigma}^z_{k-1}\hat{\sigma}^x_k\hat{\sigma}^z_{k+1}|G\rangle = |G\rangle$

Important implications for the contexts of MBL, SPT, QC, error correction …

H.J. Briegel, R Raussendorf, PRL (2001) ; M. Hein, et al., PRA (2004); R. Raussendorf et al., PRA (2003);
M.A. Nielson, Rep. Math. Phys. (2006) ; E Altman, A. Vishwanath et al., Nat Comm (2015); …

Topological nature of the cluster state

Topological zero energy edge modes：

$$\tau_x = \sigma_1{}^z \sigma_2{}^z, \tau_y = \sigma_1{}^y \sigma_2{}^z, \tau_z = \sigma_1{}^z$$ spin-1/2 edge modes

String order parameter:

$$O_{\mathrm{st}}(i,j) = \langle \sigma_i^z \sigma_{i+1}^y \left(\prod_{k=i+2}^{j-2} \sigma_k^x\right) \sigma_{j-1}^y \sigma_j^z \rangle$$

A. Vishwanath et al., Nat Comm (2015)

Although the model is only solvable in the particular stabilizer code limit, the ground state remains topologically protected against perturbations assuming the symmetry

Exact representation of the cluster state

Stabilizer ground state: $$\hat{\sigma}^z_{k-1}\hat{\sigma}^x_k\hat{\sigma}^z_{k+1}|G\rangle = |G\rangle$$

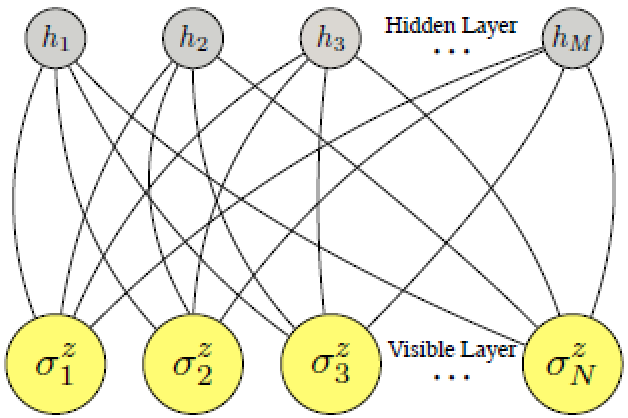
QNN wavefunction:

$$\Psi_M(\mathcal{S};\mathcal{W}) = \sum_{\{h_i\}} e^{\sum_j a_j \sigma_j^z + \sum_i b_i h_i + \sum_{ij} W_{ij} h_i \sigma_j^z}$$

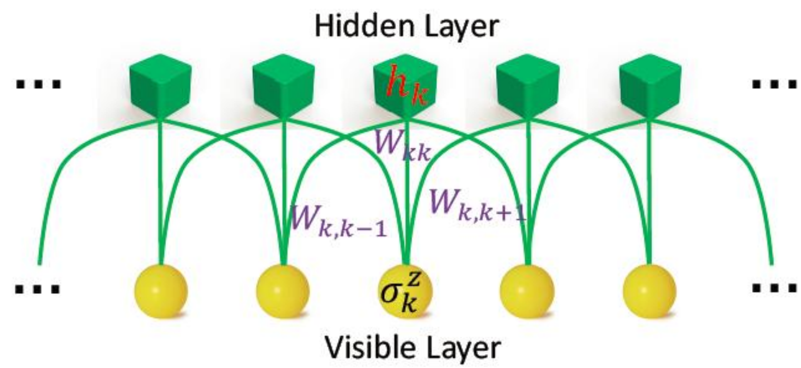
The representation to do:

$$\left|G\right\rangle = \sum_{\{S\}} \psi(S;W)\left|s_1 s_2 \ldots s_N\right\rangle$$

Exact representation of the cluster state



G. Carleo and M. Troyer, arXiv: 1606.02318 (2016)



Dong-Ling Deng, XL, S. Das Sarma, arXiv: 1609.09060(2016)

A further-restricted restricted Boltzmann machine

Exact representation of the cluster state

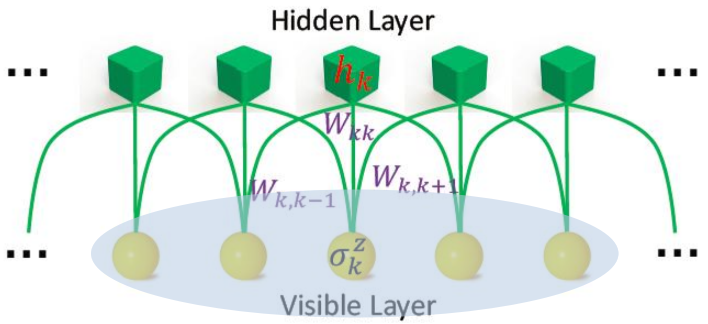
$$\left|G\right\rangle = \sum_{\{S\}} \psi(S;W) \left| s_1 s_2 \ldots s_N \right\rangle$$

$$\sigma_{k-1}^{z} \sigma_{k}^{x} \sigma_{k+1}^{z} \sum_{\{S\}} \psi(S;W) \left| s_1 s_2 \ldots s_N \right\rangle = \sum_{\{S\}} \psi(S;W) \left| s_1 s_2 \ldots s_N \right\rangle$$

$$\sum_{\{S\}} \psi(S;W) \left[s_{k-1} s_{k+1} \right] \left| s_1 s_2 \ldots s_N \right\rangle = \sum_{\{S\}} \psi(S;W) \left| s_1 s_2 \ldots (-s_k) \ldots s_N \right\rangle$$

Symmetry constraints: $a_k = ia, b_k = ib, W_{kk} = \omega_0, W_{kk\pm 1} = \omega_{\pm 1}$

We have 5 variables but 32 equations, so we do not necessarily have a solution. But we do have found one!



*the restricted nature of RBM is important to reduce the equations

Exact representation of the cluster state

A compact form of cluster state with QNN representation

$$\psi(S;W)=\sum_{h}\exp\left\{\frac{i\pi}{4}\sum_{k}h_k(1+2s_{k-1}+3s_k+s_{k+1})\right\}$$

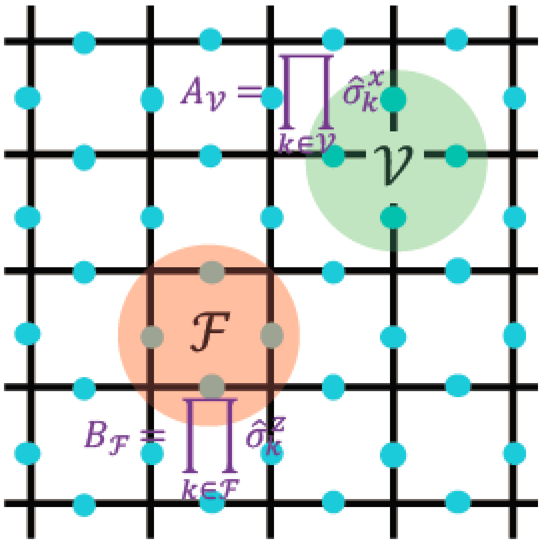
Remark: The solution may not be unique

•Demonstrated the power of QNN in representing SPT phases
•Supervise numeric machine learning SPT phases in non-exact solvable models
•Read off entanglement properties

Dong-Ling Deng, XL, S. Das Sarma, arXiv (2016)

2d Toric model

$$H_{\text{Kitaev}} = -\sum_{\mathcal{V}\in\mathbb{T}^2} A_{\mathcal{V}} - \sum_{\mathcal{F}\in\mathbb{T}^2} B_{\mathcal{F}}$$



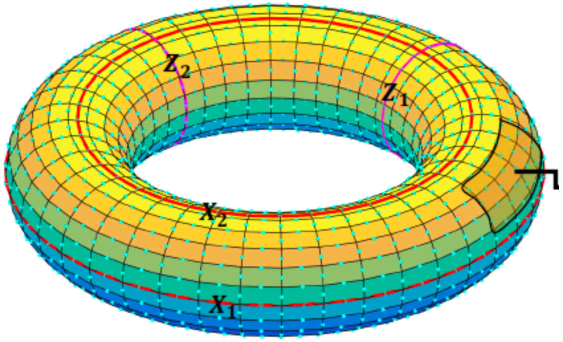
$$[A_V, A_{V'}]_{V\neq V'} = 0 \quad [B_F, B_{F'}]_{F\neq F'} = 0 \quad [A_V, B_F] = 0$$

$$A_{\mathcal{V}}|G_{\text{toric}}\rangle = |G_{\text{toric}}\rangle$$
$$B_{\mathcal{F}}|G_{\text{toric}}\rangle = |G_{\text{toric}}\rangle$$

stabilizer ground state

Topological nature of Toric code

-ground state degeneracy on a torus



X and Z encode the two topological qubits

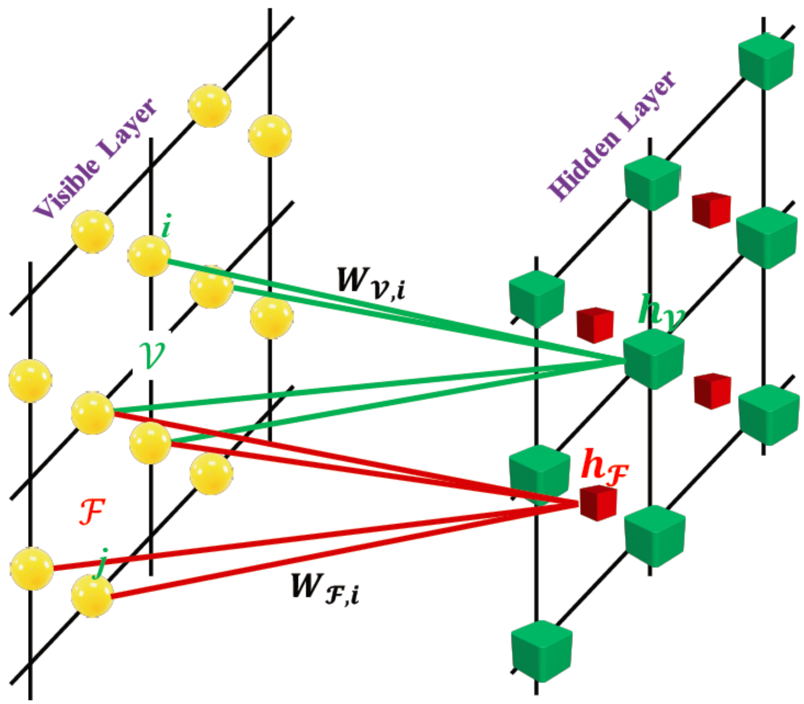
-anyons

$$e\times e=1; m\times m=1; e\times m=\psi$$

true topological order

……

Exact QNN representation of Toric code



$$\psi(S,W) = \sum_{\{h_v, h_F\}} \exp\left\{\sum_{k,V} W_{Vk}^{(1)} h_V \sigma_k^z + \sum_{k,F} W_{Fk}^{(2)} h_F \sigma_k^z\right\}$$

more difficult than 1d

Dong-Ling Deng, XL, S. Das Sarma, arXiv (2016)

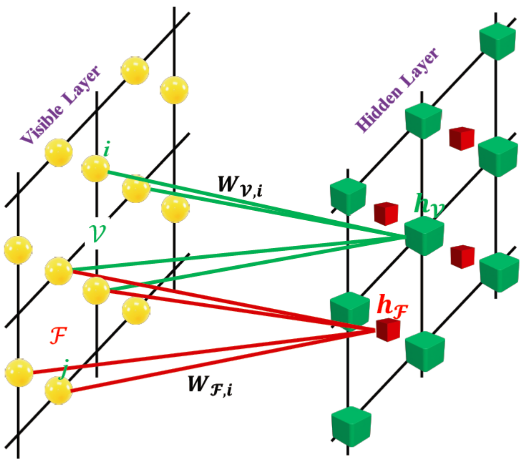
Exact QNN representation of Toric code

Simply the problem by summing over hidden neurons

$$\psi(S,W) = \prod_V \Gamma_V \prod_F \Gamma_F$$

$$\Gamma_V = 2\cosh[\sum_k W_{Vk}{}^{(1)} \sigma_k^z]$$

$$\Gamma_F = 2\cosh[\sum_k W_{Fk}{}^{(2)} \sigma_k^z]$$



Easy one: $B_{\mathcal{F}}|G_{\text{toric}}\rangle = |G_{\text{toric}}\rangle$ no spin flip

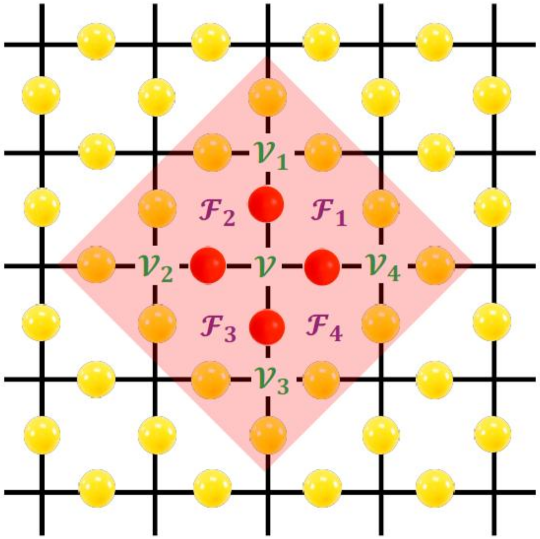
$$\longrightarrow W_{Fk}^{(2)} = \frac{i\pi}{4}$$

Difficult one: $A_{\mathcal{V}}|G_{\text{toric}}\rangle = |G_{\text{toric}}\rangle$ spin flip causes difficulty

Exact QNN representation of Toric code

$$\Gamma_V \prod_{\mu=1}^{4} \Gamma_{V_\mu} \Gamma_{F_\mu}$$
$$= \Gamma_V (\sigma_j^z \to -\sigma_j^z, \forall j \in V)$$
$$\times \prod_{\mu=1}^{4} \Gamma_{V_\mu} (\sigma_j^z \to -\sigma_j^z, \forall j \in V) \Gamma_{F_\mu} (\sigma_j^z \to -\sigma_j^z, \forall j \in V)$$

$$\Gamma_V = 2\cosh[\sum_k W_{Vk}{}^{(1)} \sigma_k^z]$$
$$\Gamma_F = 2\cosh[\sum_k W_{Fk}{}^{(2)} \sigma_k^z]$$



We now have 4 variables and 65536 equations. Much more difficult to deal with than 1d. 3d would be even more difficult. Actually we did not work out any 3d models.

$$\psi(S,W) = \sum_{\{h_V, h_F\}} \exp\left\{ \frac{i\pi}{2} \sum_{k,V} h_V \sigma_k^z + \frac{i\pi}{4} \sum_{k,F} h_F \sigma_k^z \right\}$$

What we gain:

- Demonstrated the power of QNN in representing topological orders despite the long-range entanglement
- Supervise numeric machine learning 2d topological orders in non-exact solvable models

Summary

- Restricted Boltzmann machine and quantum generalization
- 1d graph state (SPT) and exact quantum neuron network representation
- 2d toric code state (topological order) and exact quantum neuron network representation

Dong-Ling Deng, XL, S. Das Sarma, arXiv 1609.09060 (2016)